

## SHEAFIFICATIONS OF THE DE RHAM-WITT COMPLEX

VERONIKA ERTL AND LANCE EDWARD MILLER

**ABSTRACT.** Recent important and powerful frameworks for the study of differential forms by Huber-Jörder and Huber-Kebekus-Kelly based on Voevodsky's h-topology have greatly simplified and unified many approaches. This article builds towards the goal of putting Illusie's de Rham-Witt complex in the same framework by exploring the h-sheafification of the rational de Rham-Witt differentials. Assuming resolution of singularities in positive characteristic one recovers a complete cohomological h-descent for all terms of the complex. We also provide unconditional h-descent for the global sections and draw the expected conclusions. The approach is to realize that a certain right Kan extension introduced by Huber-Kebekus-Kelly takes the sheaf of rational de Rham-Witt forms to a qfh-sheaf. As such, we state and prove many results about qfh-sheaves which are of independent interest.

## 1. INTRODUCTION

Among the most ubiquitous tools for studying varieties over a field are differential forms, or more formally de Rham cohomology and its Hodge type variants. These are well-known to only be well-behaved for smooth varieties over the complex numbers or other algebraically closed fields of characteristic 0; the key facet being Hironaka's resolution of singularities. Adding to the difficulty in extending these methods, even defining such forms outside the smooth case becomes a challenge, with many variants tailored to specific considerations, e.g., reflexive differentials for the normal setting, log differentials, the Du Bois complex etc. However, recent work of Huber-Jörder [HJ14] and Huber-Kebekus-Kelly [HKK15] have described a successful and powerful approach unifying and conceptualizing studies using differential forms by introducing sheaf theoretic methods utilizing various Grothendieck topologies, notably Voevodsky's h-topology. This approach is built on the intuition that all varieties over a field of characteristic 0 are h-locally smooth. The h-topology meets challenges in positive characteristic, where one quickly notes that the basic Kähler differentials become zero under h-sheafification owing to the Frobenius map being an h-cover. However, more subtle sheaf methods help to overcome this difficulty via a specific right Kan extension.

In  $p$ -adic Hodge theory, differential forms are rarely themselves the right consideration and one notes the ubiquity of Illusie's de Rham-Witt complex and its variants. The aim of this article is to consider these complexes and their sheafifications under the same Grothendieck topologies considered in [HJ14] and [HKK15]. Specifically, we consider h-descent statements for the differentials  $W\Omega_{\mathbf{Q}}^n = W\Omega^n \otimes \mathbf{Q}$  of the rational de Rham-Witt complex. An optimal result would be a cohomological descent statement analogous to [HJ14, Cor. 6.5]. However, we can only recover it up to an assumption of resolution of singularities in positive characteristic. In particular, we describe a proof of the following theorem, which is well-known to experts.

**Theorem.** *Let  $X$  be smooth variety over a perfect field of characteristic  $p$ . Assume that resolution of singularities holds. Consider the h-sheaf and  $W\Omega_{\mathbf{Q},h}^n$ . For all  $i \geq 0$  and  $n \geq 0$*

$$H_h^i(X, W\Omega_{\mathbf{Q},h}^n) \cong H_{\text{Zar}}^i(X, W\Omega_{\mathbf{Q}}^n).$$

Avoiding resolution of singularities presents, at present, an insurmountable challenge. Undeterred, we exploit many of the techniques of [HKK15] to recover a descent statement for all  $n$  and  $i = 0$  free of any assumption on resolution of singularities. This relies strongly on considering the  $(-)_{\text{dvr}}$ -construction, which is related to the h-sheafification. In particular, for a base scheme  $S$ , denote by  $\text{Sch}(S)$  the category of separated finite type  $S$ -schemes and  $\text{Reg}(S)$  the full subcategory of regular schemes. The restriction  $\text{Sch}(S) \rightarrow \text{Reg}(S)$  has a right adjoint  $(-)_{\text{dvr}}$  on the category of presheaves, and this auxiliary sheaf is critical to extending the results of [HJ14] to the de Rham-Witt setting. The biggest new result of the paper shows that this presheaf is a sheaf for the quasi-finite or qfh-topology.

**Theorem.** *Let  $k$  be a perfect field of characteristic  $p > 0$ . The presheaf  $W\Omega_{\mathbf{Q},\text{dvr}}^n$  is a qfh-sheaf on  $\text{Sch}(k)$ .*

The key point is that under expected circumstances, for sheaves such as these one may establish an h-descent.

**Theorem.** *Fix a base scheme  $S$  and a qfh-sheaf  $\mathcal{F}$  on  $\text{Sch}(S)$ . When  $\mathcal{F}$  is also an eh-sheaf,  $\mathcal{F}_{\text{dvr}}$  satisfies h-descent on  $\text{Sch}(S)$ . In particular, for a perfect field of characteristic  $p$ , the sheaves  $W\Omega_{\mathbf{Q},\text{dvr}}^n$  on  $\text{Sch}(k)$  satisfy h-descent.*

We draw from this many expected consequences, in particular recovering an analogue of [HJ14, Prop. 4.2] for the rational de Rham-Witt differentials.

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## 2. PRELIMINARIES AND NOTATIONAL CONVENTIONS

Throughout this work we consider Grothendieck topologies. We refer to [SP] as a general reference for the uninitiated reader. All schemes are assumed to be separated. For a fixed noetherian base scheme  $S$  denote by  $\mathrm{Sch}(S)$  the category of separated schemes of finite type over  $S$ . When  $S = \mathrm{Spec} R$  is affine, we conflate notation writing  $\mathrm{Sch}(R)$  for  $\mathrm{Sch}(S)$ . We denote by  $\mathrm{Reg}(S)$  the full subcategory of  $\mathrm{Sch}(S)$  of regular schemes, by  $\mathrm{Nor}(S)$  the full subcategory of  $\mathrm{Sch}(S)$  of normal schemes, and by  $\mathrm{Var}(S)$  the full subcategory of  $\mathrm{Sch}(S)$  which are reduced and irreducible. For a Grothendieck topology  $\tau$  on  $\mathrm{Sch}(S)$  we denote by  $\mathrm{Sch}(S)_\tau$  the associated site and by  $\mathrm{Sh}(S)_\tau$  the associated topos.

Fix a base  $S$ . It will be critically important to distinguish various sheafifications involving complexes of sheaves on  $\mathrm{Sch}(S)$ . Consider  $\tau$  a topology on the category  $\mathrm{Sch}(S)$ . For a presheaf  $\mathcal{F}$  on  $\mathrm{Sch}(S)$ , denote by  $\mathcal{F}_\tau$  the sheafification of  $\mathcal{F}$ . Via functoriality, for a complex  $C^\bullet$  of presheaves, the sheafifications of the terms can be placed into a new complex. We denote by  $C_\tau^\bullet$  the complex with terms  $C_\tau^n$ . We note that this is not the same thing as considering a  $\tau$ -sheaf valued in a category of complexes and often contains less information. Despite this, it carries quite a bit of information and was the setting considered in [HJ14] which formed the inspiration for this work.

**Topologies.** Fix a category  $\mathcal{C}$ . For two Grothendieck topologies  $\tau$  and  $\mu$  on  $\mathcal{C}$  one calls  $\tau$  **finer** than  $\mu$  provided every  $\mu$ -cover is a  $\tau$ -cover. This may be denoted  $\mu \rightarrow \tau$  and it induces a morphism of sites  $\mathcal{C}_\tau \rightarrow \mathcal{C}_\mu$ . In particular, for any fixed base  $S$  and topology  $\tau$  on  $\mathrm{Sch}(S)$  finer than the Zariski topology, one has an associated morphism of sites  $\rho: \mathrm{Sch}(S)_\tau \rightarrow \mathrm{Sch}(S)_{\mathrm{Zar}}$  inducing a geometric morphism of topoi  $(\rho^*, \rho_*): \mathrm{Sh}(S)_\tau \rightarrow \mathrm{Sh}(S)_{\mathrm{Zar}}$ . These also have extensions to morphisms of bounded derived categories  $D^+(\mathrm{Sh}(S)_\tau) \rightarrow D^+(\mathrm{Sh}(S)_{\mathrm{Zar}})$ . Thus one may consider  $R\rho_*(C^\bullet)$  of any complex  $C^\bullet \in D^+(\mathrm{Sh}(S)_\tau)$ .

Fix a base scheme  $S$  and two topologies  $\tau$  and  $\mu$  on  $\mathrm{Sch}(S)$  with  $\mu \rightarrow \tau$ . For  $X \in \mathrm{Sch}(S)$ , we say that a  $\tau$ -sheaf  $\mathcal{F}$  satisfies  **$\mu$ -descent for  $X$**  provided that one has an isomorphism  $\mathcal{F}_\mu(X) \cong \mathcal{F}_\tau(X)$  and that  $\mathcal{F}$  satisfies **cohomological  $\mu$ -descent for  $X$**  provided  $H_\mu^i(X, \mathcal{F}_\mu) \cong H_\tau^i(X, \mathcal{F})$  for all  $i \geq 0$ .

We are primarily concerned with Voevodsky's h-topology (cf. [Voe96]) in particular the characteristic  $p$  case. An overview geared towards the mixed characteristic setting can also be found in [Bei12]. We review the basic definition.

**Definition 2.1.** A morphism in  $\mathrm{Sch}(S)$  is called a **topological epimorphism**, if it is surjective and the Zariski topology of the target is the quotient topology of the Zariski topology of the source. It is called **universal** if this property is preserved by any base change. The **h-topology** is the Grothendieck topology on  $\mathrm{Sch}(S)$  with coverings universal topological epimorphisms of finite type.

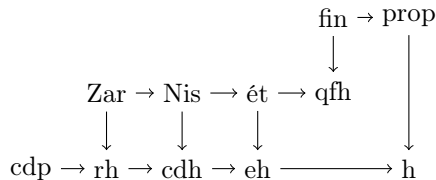
The h-topology is the coarsest topology finer than the proper and Zariski topology. It is also finer than the étale topology and in fact it is finer than the eh-topology which is generated by étale covers and so called abstract blow-ups.

**Definition 2.2.** Let  $X \in \mathrm{Sch}(S)$ ,  $f: X' \rightarrow X$  a proper map and  $Z \subset X$  a closed subscheme with preimage  $E \subset X'$ . We say  $(X', Z)$  is an **abstract blow-up of  $X$** , if  $f$  induces an isomorphism  $X' \setminus E \xrightarrow{\sim} X \setminus Z$  outside  $Z$ . We denote an abstract blow-up either by  $(X' \xrightarrow{f} X, Z \xrightarrow{i} X)$  or more simply by  $(X', Z)$ .

This means all proper birational maps are eh-covers and so h-covers. Moreover, finite maps are also h-covers. As such, Beilinson notes that in the presence of resolution of singularities, schemes are h-locally smooth [Bei14, Section 2] and in more general settings de Jong's alteration theorem gives that schemes over discrete valuations rings are h-locally regular.

There are several close useful relatives of the h-topology. In particular, to minor extents we will utilize the Nisnevich topology generated by completely decomposed families of étale morphisms, the cdp or envelope-topology generated by completely decomposed families of proper morphisms, the rh-topology generated by Zariski, and cdp-covers, and the cdh-topology generated by Nisnevich covers and rh-covers. A much more central topology for our consideration is the quasi-finite topology, or qfh-topology generated by étale covers and finite covers. Throughout, it will be helpful to keep in mind the following diagram of how many of these different topologies on  $\mathrm{Sch}(S)$  relate (c.f., [GK15]).

Voevodsky describes a particularly nice decomposition of h-covers which will play a critical role in the proof of Lemma 4.10. An h-cover  $\{U_i \xrightarrow{p_i} X\}$  is said to be of **normal form** when it admits a factorization  $p_i = s \circ f \circ \iota_i$ ,

FIGURE 1. Topologies on  $\text{Sch}$  from coarser to finer.

where  $\{U_i \xrightarrow{\iota_i} \overline{U}\}$  is a Zariski cover,  $\overline{U} \xrightarrow{f} X_Z$  is finite surjective and  $X_Z \xrightarrow{s} X$  is the blow-up along a closed subscheme  $Z$  of  $X$ . In relatively mild cases, all h-covers admit refinements of normal form.

**Theorem 2.3** (V. Voevodsky). *Let  $\{U_i \xrightarrow{p_i} X\}$  be an h-cover of an excellent, noetherian, reduced scheme. There exists an h-cover of normal form which is a refinement of  $\{p_i\}$ .*

**Review of differential forms.** The work of Huber–Jörder and Huber–Kebekus–Kelly describe descent for differential forms. Fix a field  $k$  and the base scheme  $S = \text{Spec}(k)$ . Denote by  $\Omega^1$  the presheaf of Kähler differential forms on  $\text{Sch}$  and for each positive integer  $q$  denote by  $\Omega^n$  its  $n$ -fold exterior power. Assume  $k$  has characteristic 0. For the relationships  $\text{cdh} \rightarrow \text{eh} \rightarrow \text{h}$  between the topologies, we have descent  $\Omega_{\text{eh}}^n(X) \cong \Omega_{\text{cdh}}^n(X)$  [HJ14, Cor. 2.8] and  $\Omega_{\text{eh}}^n(X) \cong \Omega_{\text{h}}^n(X)$  [HJ14, Thm. 3.6] for any  $X \in \text{Var}(k)$ . If  $X$  is in addition smooth, one has the descent isomorphisms  $\Omega^n(X) \cong \Omega_{\text{eh}}^n(X)$  by [Gei06, Thm. 4.7] which depends on resolution of singularities. Thus in characteristic 0 the h-sheafification of Kähler differentials agrees with the usual ones in the smooth case. For a klt-base  $X$ , one also has  $\Omega_{\text{h}}^n(X) \cong (\Omega^n(X))^{**}$  the pushforward of the differentials on the smooth locus [HJ14, Thm. 5.4].

When  $k$  is of characteristic  $p > 0$  the situation is more complicated. For example an h-sheafification  $\mathcal{F}_{\text{h}}$  of a presheaf  $\mathcal{F}$  can be zero. This happens when  $\mathcal{F}$  has  $p$ -torsion. However some descent statements can be recovered at least for the cdh-topology via an auxiliary sheaf constructed using the right adjoint of the natural restriction  $\text{Sch} \rightarrow \text{Reg}$  denoted  $(-)_{\text{dvr}}$ . Notably, when  $X \in \text{Reg}(S)$  one has  $\mathcal{F}_{\text{dvr}}(X) = \mathcal{F}(X)$  and when  $k$  has characteristic 0 one can show  $\Omega_{\text{dvr}}^n \cong \Omega_{\text{h}}^n$  using [HJ14]. This construction works particularly well under the assumption that  $\mathcal{F}$  is unramified, see [HKK15, Def. 4.5].

**The de Rham–Witt complex.** As we will primarily be concerned with questions on the de Rham–Witt complex, we set notations here. Fix a perfect field  $k$  of characteristic  $p > 0$ . For a scheme  $X$  over  $k$ , denote by  $W\Omega_X^\bullet$  the  $p$ -typical absolute de Rham–Witt complex of  $X$  defined by Illusie [Ill79]. Set  $W\Omega_{X,\mathbf{Q}}^\bullet := W\Omega_X^\bullet \otimes \mathbf{Q}$ . It is a complex of étale sheaves. For a fixed integer  $n$ , consider the presheaf  $X \mapsto W\Omega_{\mathbf{Q}}^n(X)$  of degree  $n$  differentials of the de Rham–Witt complex. We abuse terminology and refer to this as the sheaf of rational Witt differentials. For a topology  $\tau$  on  $\text{Sch}(k)$ , finer than the étale topology we denote by  $W\Omega_{\mathbf{Q},\tau}^\bullet$  the complex of  $\tau$ -sheaves whose  $n$ -th term is  $W\Omega_{\mathbf{Q},\tau}^n$ , which we call  $\tau$ -Witt differentials.

### 3. COHOMOLOGICAL DESCENT FOR DE RHAM–WITT COMPLEXES ASSUMING RESOLUTION OF SINGULARITIES

Our primary interest is in the sheaf  $W\Omega_{\mathbf{Q}}^n$ , in particular in its descent properties. For smooth schemes one hopes to establish a cohomological descent for the h-topology, i.e., isomorphisms  $H_{\text{h}}^i(X, W\Omega_{\mathbf{Q},\text{h}}^n) \cong H_{\text{Zar}}^i(X, W\Omega_{\mathbf{Q},\text{h}}^n)$  for each  $n \geq 0$ ,  $i \geq 0$  and  $X \in \text{Sm}(k)$ . This is rather easy provided one has resolution of singularities of schemes in positive characteristic and we warm up with a proof of this which is well-known to experts. Huber and Jörder in [HJ14, Prop. 6.1] give an h-descent for eh-sheaves of  $\mathbf{Q}$ -vector spaces which directly applies to our context. Assuming resolution of singularities, an argument of Geisser [Gei06, Thm. 4.2] goes through for  $W\Omega_{\mathbf{Q},\text{h}}^n$  giving an eh-descent for  $W\Omega_{\mathbf{Q},\text{h}}^n$ . This suffices to establish the cohomological descent via the usual cohomological étale descent for de Rham–Witt sheaves. The key point is that factorization of h-covers can be done through smooth blow-ups. Throughout this section, we work with schemes over a perfect field  $k$  of characteristic  $p > 0$ . A needed ingredient is the following well-known lemma.

**Lemma 3.1.** [Blow-up sequence] *Let  $(X' \xrightarrow{f} X, Z \xrightarrow{i} X)$  be an abstract blow-up of  $k$ -schemes, and  $\mathcal{F}$  an h-sheaf on  $\text{Sch}(k)$ . The associated blow-up sequence*

$$\cdots \rightarrow H_{\text{h}}^i(X, \mathcal{F}) \rightarrow H_{\text{h}}^i(X', \mathcal{F}) \oplus H_{\text{h}}^i(Z, \mathcal{F}) \rightarrow H_{\text{h}}^i(Z', \mathcal{F}) \rightarrow \cdots$$

*is exact.*

*Proof.* In [Gei06, Prop. 3.2], Geisser proves the analogous statement for the eh-topology in characteristic  $p$ . Notably one does not need to assume resolution of singularities. Since every eh-cover is also an h-cover, it follows that the statement holds also for the h-topology. Thus, the result follows by replacing h with eh in Geisser's proof.  $\square$

**Theorem 3.2.** *Assume that resolution of singularities holds. Let  $X$  be smooth  $k$ -scheme. Consider the h-sheaf  $W\Omega_{\mathbf{Q},h}^n$ . For all  $i \geq 0$  and  $n \geq 0$*

$$H_h^i(X, W\Omega_{\mathbf{Q},h}^n) \cong H_{\text{Zar}}^i(X, W\Omega_{\mathbf{Q}}^n).$$

*Proof.* First we note that as in the proof of [HJ14, Prop. 6.1], which does not depend on the characteristic of the base field, one deduces the equality  $H_h^i(X, W\Omega_{\mathbf{Q},h}^n) \cong H_{\text{eh}}^i(X, W\Omega_{\mathbf{Q},\text{eh}}^n)$ .

To obtain the equality  $H_{\text{eh}}^i(X, W\Omega_{\mathbf{Q},\text{eh}}^n) \cong H_{\text{ét}}^i(X, W\Omega_{\mathbf{Q}}^n)$  the strategy of [Gei06, Thm. 4.3] can be applied. Let  $\rho : \text{Sch}(k)_{\text{eh}} \rightarrow \text{Sch}(k)_{\text{ét}}$  be the canonical morphism of sites, which induces a morphism of étale sheaves  $W\Omega_{\mathbf{Q}}^n \rightarrow R\rho_* W\Omega_{\mathbf{Q},\text{eh}}^n$ . Let  $C^\bullet$  be the cone of this map. One shows by contradiction that  $H_{\text{ét}}^i(Y, C^\bullet) = 0$  for any  $Y \in \text{Sm}(k)$ . Assume that there is  $0 \neq y \in H_{\text{ét}}^i(Y, C^\bullet)$  and that moreover  $Y \in \text{Sm}(k)$  is of smallest dimension admitting such a non-zero element. It is  $\rho^* C^\bullet = 0$  by definition of  $C^\bullet$ , which means that there is an eh-cover  $\mathfrak{U}_\bullet$  of  $Y$ , such that  $C^\bullet|_{\mathfrak{U}_\bullet} = 0$ . According to [Gei06, Corollary 2.6]  $\mathfrak{U}_\bullet$  has, under resolution of singularities a refinement  $\{U_i \rightarrow Y' \rightarrow Y\}$ , where  $\{U_i \rightarrow Y'\}$  is an étale cover and  $Y' \rightarrow Y$  is a composition of smooth blow-ups. For a given blow-up  $Y' \rightarrow Y$  along a smooth centre  $Z$ , we have by Lemma 3.1 the blow-up sequence

$$\cdots \rightarrow H_{\text{eh}}^i(Y, W\Omega_{\mathbf{Q},\text{eh}}^n) \rightarrow H_{\text{eh}}^i(Y', W\Omega_{\mathbf{Q},\text{eh}}^n) \oplus H_{\text{eh}}^i(Z, W\Omega_{\mathbf{Q},\text{eh}}^n) \rightarrow H_{\text{eh}}^i(Z', W\Omega_{\mathbf{Q},\text{eh}}^n) \rightarrow \cdots$$

On the other hand, a result by Gros [Gro85, IV. Théorème 1.1.9] gives the analogous sequence for étale topology

$$\cdots \rightarrow H_{\text{ét}}^i(Y, W\Omega_{\mathbf{Q}}^n) \rightarrow H_{\text{ét}}^i(Y', W\Omega_{\mathbf{Q}}^n) \oplus H_{\text{ét}}^i(Z, W\Omega_{\mathbf{Q}}^n) \rightarrow H_{\text{ét}}^i(Z', W\Omega_{\mathbf{Q}}^n) \rightarrow \cdots$$

The canonical map induces a morphism of long exact sequences

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_{\text{ét}}^i(Y, W\Omega_{\mathbf{Q}}^n) & \longrightarrow & H_{\text{ét}}^i(Y', W\Omega_{\mathbf{Q}}^n) \oplus H_{\text{ét}}^i(Z, W\Omega_{\mathbf{Q}}^n) & \longrightarrow & H_{\text{ét}}^i(Z', W\Omega_{\mathbf{Q}}^n) \longrightarrow \cdots \\ & & \tau_Y \downarrow & & \tau_{Y'} \downarrow \tau_Z & & \tau_{Z'} \downarrow \\ \cdots & \longrightarrow & H_{\text{eh}}^i(Y, W\Omega_{\mathbf{Q},\text{eh}}^n) & \longrightarrow & H_{\text{eh}}^i(Y', W\Omega_{\mathbf{Q},\text{eh}}^n) \oplus H_{\text{eh}}^i(Z, W\Omega_{\mathbf{Q},\text{eh}}^n) & \longrightarrow & H_{\text{eh}}^i(Z', W\Omega_{\mathbf{Q},\text{eh}}^n) \longrightarrow \cdots \end{array}$$

By minimality of  $Y$  the maps  $\tau_Z$  and  $\tau_{Z'}$  are isomorphisms, from which one concludes that  $0 \neq y|_{Y'} \in H^i(Y', C^\bullet)$ . In particular,  $C^\bullet|_{Y'}$  is not acyclic. But then it is also not acyclic on the étale cover  $\{U_i \rightarrow Y'\}$ , a contradiction to  $\rho^* C^\bullet = 0$ .

Finally, it is well known that in the case at hand étale and Zariski cohomology coincide, which together with the above equalities leads to the desired statement.  $\square$

We cannot yet establish (smooth) cohomological h-descent for  $W\Omega_{\mathbf{Q}}^n$  without assuming resolution of singularities. The difficulty lies in the reliance on Geisser's work. One is tempted to utilize de Jong's alterations as a replacement but technical questions still abound. As we have seen in the proof above, Geisser uses the refinement of an eh-cover in an étale cover and a composition of smooth blow-ups. To obtain such a refinement strong resolution of singularities is needed. If one attempts to use alterations here instead finite maps are introduced in the process, which cannot be controlled. Instead, we utilize ideas from [HKK15] to establish h-descent statements avoiding resolutions of singularities by showing an h-descent for an auxiliary sheaf  $W\Omega_{\mathbf{Q},\text{dvr}}^n$  which agrees with  $W\Omega_{\mathbf{Q}}^n$  in the regular case.

#### 4. dvr-WITT DIFFERENTIALS

Recall, for a base scheme  $S$  the restriction  $\text{Sch}(S) \rightarrow \text{Reg}(S)$  has a right adjoint  $(\_)_{\text{dvr}}$  on the category of presheaves, called the extension functor. This functor may be understood using schemes essentially of finite type in the sense of [HKK15, Def. 2.5]. Let  $\text{Sch}(S)^{\text{ess}}$  be the category of schemes essentially of finite type, which contains  $\text{Sch}(S)$  as a subcategory, and  $\text{Dvr}(S)$  the category of schemes essentially of finite type which are regular, local, and of dimension at most 1. A presheaf  $\mathcal{F}$  on  $\text{Sch}(S)$  can be extended to a presheaf  $\mathcal{F}^{\text{ess}}$  on  $\text{Sch}(S)^{\text{ess}}$  as explained in [HKK15, Def./Prop. 4.9]. In many cases of interest,  $\mathcal{F}$  and  $\mathcal{F}^{\text{ess}}$  coincide on  $\text{Sch}(S)^{\text{ess}}$ . As described in [HKK15], torsion is an important concept in the present context and will be treated in Subsection 4.3.

**4.1. The extension functor applied to Witt differentials.** We summarize the extension functor  $(\_)\_{\text{dvr}}$  and explore results for the rational de Rham-Witt complex. For more details on the extension functor see [HKK15, Sec. 4.1]. Fix a base scheme  $S$ .

**Definition 4.1.** For a presheaf  $\mathcal{F}$  on  $\text{Reg}(S)$ , let  $\mathcal{F}_{\text{dvr}}$  be the presheaf on  $\text{Sch}(S)$  given by

$$\mathcal{F}_{\text{dvr}}(X) := \varprojlim_{Y \in \text{Reg}(X)} \mathcal{F}(Y)$$

for  $X \in \text{Sch}(S)$ .

Huber, Kebekus and Kelly show that  $\mathcal{F}_{\text{dvr}}$  can be expressed in a particularly useful form. This requires the notion of an unramified sheaf as introduced by Morel, [HKK15, Def. 4.5].

**Definition 4.2.** A presheaf  $\mathcal{F}$  on  $\text{Sch}(S)$  is **unramified** provided it is a Zariski sheaf and satisfies the following three conditions for all  $X$  and  $Y$  in  $\text{Reg}(S)$

- (i) the natural morphism  $\mathcal{F}(X \sqcup Y) \rightarrow \mathcal{F}(X) \times \mathcal{F}(Y)$  is an isomorphism,
- (ii) For  $U \rightarrow X$  a dense open immersion,  $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$  is injective,
- (iii) for every open immersion  $U \rightarrow X$  containing all points of codimension at most 1,  $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$  is an isomorphism.

If  $X \in \text{Sch}(S)$  and  $\mathcal{F}$  is unramified, then [HKK15, Prop 4.14] guarantees that

$$\mathcal{F}_{\text{dvr}}(X) = \varprojlim_{W \in \text{Dvr}(X)} \mathcal{F}^{\text{ess}}(W). \quad (4.1)$$

We set from now on  $S = \text{Spec } k$ , where  $k$  is a perfect field of characteristic  $p > 0$  as we now want to apply this technology to study descent for the de Rham-Witt sheaves  $W\Omega_{\mathbf{Q}}^n$  where  $n \in \mathbf{N}$ .

**Theorem 4.3.** Both the presheaf  $W\Omega^n$  and the presheaf  $W\Omega_{\mathbf{Q}}^n$  are unramified on  $\text{Reg}(k)$ .

*Proof.* It suffices to show that  $W\Omega^n$  is unramified. We have to verify the three conditions defining unramified presheaves. As we are working with regular schemes of finite type, we can take advantage of functoriality properties of the de Rham-Witt complex. Specifically, the structure sheaf  $\mathcal{O}$  is an unramified presheaf. Note that when  $X \in \text{Reg}(k)$ , global sections of Witt differentials are given by  $\Gamma(X, W\Omega^n) = W\Omega_{\Gamma(X, \mathcal{O})}^n$ , [Ill79, I.1.10].

It is well-known, that  $W\Omega^n$  is an étale sheaf, whence a Zariski sheaf. Let  $X, Y \in \text{Reg}(k)$ . Since  $\mathcal{O}$  is unramified, one has by property (i)  $\mathcal{O}(X \amalg Y) = \mathcal{O}(X) \times \mathcal{O}(Y)$ . With this, we compute

$$W\Omega^n(X \amalg Y) = W\Omega_{\mathcal{O}(X \amalg Y)}^n = W\Omega_{\mathcal{O}(X) \times \mathcal{O}(Y)}^n = W\Omega_{\mathcal{O}(X)}^n \times W\Omega_{\mathcal{O}(Y)}^n = W\Omega^n(X) \times W\Omega^n(Y).$$

Let  $U \rightarrow X$  be a dense open immersion. The induced map  $\mathcal{O}(X) \rightarrow \mathcal{O}(U)$  is injective because  $\mathcal{O}$  is unramified. It follows by functoriality of the de Rham-Witt complex, that

$$W\Omega^n(X) = W\Omega_{\mathcal{O}(X)}^n \rightarrow W\Omega_{\mathcal{O}(U)}^n = W\Omega^n(U)$$

is also injective. Finally, let  $U \rightarrow X$  be an open immersion which contains all points of codimension at most 1. The map  $\mathcal{O}(X) \rightarrow \mathcal{O}(U)$  is then an isomorphism, and by functoriality the same holds true for the induced map  $W\Omega^n(X) \rightarrow W\Omega^n(U)$ .  $\square$

We call the presheaf  $W\Omega_{\mathbf{Q}, \text{dvr}}^n$  obtained by applying the extension functor to the presheaf  $W\Omega_{\mathbf{Q}}^n$  of rational Witt differentials on  $\text{Sch}(k)$  the **dvr-Witt differentials** of degree  $n$ .

**Corollary 4.4.** Let  $X \in \text{Sch}(k)$ . The dvr-Witt differentials of  $X$  can be written as

$$W\Omega_{\mathbf{Q}, \text{dvr}}^n(X) = \varprojlim_{Y \in \text{Dvr}(X)} W\Omega_{\mathbf{Q}}^{n, \text{ess}}(Y).$$

*Proof.* Theorem 4.3 shows that  $W\Omega_{\mathbf{Q}}^n$  is unramified. The claim then follows from equation (4.1).  $\square$

**4.2. dvr-Witt differentials and the h-topology.** The dvr-Witt differentials enjoy many properties that will help us to show h-descent for Witt differentials. A prime example is a result of Huber, Kebekus, and Kelly showing useful descent properties.

**Theorem 4.5.** (c.f., [HKK15, Prop. 4.18]) Let  $S$  be a noetherian scheme. If  $\mathcal{F}$  is an unramified presheaf on  $\text{Sch}(S)$ , then  $\mathcal{F}_{\text{dvr}}$  is an rh-sheaf. In particular, if  $\mathcal{F}$  is an unramified Nisnevich, respectively étale sheaf on  $\text{Sch}(S)$ , then  $\mathcal{F}_{\text{dvr}}$  is a cdh-sheaf, respectively eh-sheaf.

Thus for a perfect field  $k$  of positive characteristic  $p$ , the presheaf  $W\Omega_{\mathbf{Q}, \text{dvr}}^n$  is an eh-sheaf on  $\text{Sch}(k)$ .

**Remark 4.6.** *By the universal property of sheafification there are canonical morphisms*

$$W\Omega_{\mathbf{Q}}^n \rightarrow W\Omega_{\mathbf{Q},\text{rh}}^n \rightarrow W\Omega_{\mathbf{Q},\text{cdh}}^n \rightarrow W\Omega_{\mathbf{Q},\text{eh}}^n \rightarrow W\Omega_{\mathbf{Q},\text{dvr}}^n.$$

Moreover, if  $X$  is regular,  $W\Omega_{\mathbf{Q}}^n(X) = W\Omega_{\mathbf{Q},\text{dvr}}^n(X)$  by [HKK15, Rem. 4.3.3] and the above composition is an isomorphism on  $X$ .

Following an approach used in the study of regular group schemes [AHL16, Prop. C.2], we establish preliminary descent results.

**Theorem 4.7.** *For a perfect field  $k$  of characteristic  $p > 0$ , the canonical morphism  $W\Omega_{\mathbf{Q}}^n \rightarrow W\Omega_{\mathbf{Q},\text{qfh}}^n$  is an isomorphism on  $\text{Nor}(k)$ .*

*Proof.* From [HKK15, Lem. 4.4] we know that the extension functor preserves sheaves. When  $W\Omega_{\mathbf{Q}}^n$  is a qfh-sheaf on  $\text{Nor}(k)$  and therefore on  $\text{Reg}(k)$ , its extension  $W\Omega_{\mathbf{Q},\text{dvr}}^n$  is a qfh-sheaf on  $\text{Sch}(k)$ . Hence, it is enough to show that the sheaf  $W\Omega_{\mathbf{Q}}^n$  is a qfh-sheaf on  $\text{Nor}(k)$ , for which we show the canonical morphism  $W\Omega_{\mathbf{Q}}^n \rightarrow W\Omega_{\mathbf{Q},\text{qfh}}^n$  induces an isomorphism on normal schemes.

To show that  $W\Omega_{\mathbf{Q}}^n$  is a qfh-sheaf on  $\text{Nor}(k)$  we check the sheaf condition for a qfh-cover  $\{U_i \rightarrow X\}_{i \in I}$  of normal schemes. Without loss of generality, we may assume that  $X$  is connected and therefore integral as it is noetherian. By a result of Suslin–Voevodsky [SV96, Lem. 10.4] the qfh-cover above has a refinement of the form  $\{V_j \rightarrow V \rightarrow X\}_{j \in J}$ , where  $V$  is the normalization of  $X$  in a finite normal extension of its function field. In particular,  $V \rightarrow X$  is finite surjective and  $\{V_j \rightarrow V\}$  is a Zariski cover.

The sheaf condition for  $\{V_j \rightarrow X\}$  is implied by the sheaf condition for  $\{V_j \rightarrow V\}$  and  $V \rightarrow X$  separately. To see this we argue as in [AHL16] that  $W\Omega_{\mathbf{Q}}^n|_{\text{Nor}}$  is separated for the qfh-topology. This follows from the fact that for a dominant morphism  $f : X \rightarrow Y$  the induced morphism  $W\Omega_{\mathbf{Q}}^n(Y) \rightarrow W\Omega_{\mathbf{Q}}^n(X)$  is injective, as  $\mathcal{O}(Y) \rightarrow \mathcal{O}(X)$  is and using a result due to Voevodsky [Voe96, Prop. 3.1.4].

We know already that  $W\Omega_{\mathbf{Q}}^n$  satisfies the sheaf condition for a Zariski cover  $\{V_j \rightarrow V\}$ . Hence it remains to show the sheaf condition for a finite surjective morphism  $\pi : V \rightarrow X$  of normal irreducible schemes.

Let  $\tilde{V}$  be the normalization of the reduction of  $V \times_X V$  and consider the sequence

$$0 \rightarrow W\Omega_{\mathbf{Q}}^n(X) \rightarrow W\Omega_{\mathbf{Q}}^n(V) \rightarrow W\Omega_{\mathbf{Q}}^n(\tilde{V})$$

where the rightmost map is induced by the difference of the two projections  $\tilde{V} \rightarrow V$ . Our objective is to show this sequence is exact.

We may find a finite surjective morphism  $\pi' : Y \rightarrow V$  with  $Y$  normal, such that the composition factors into a generically purely inseparable morphism followed by a generically Galois morphism  $\pi'\pi = \pi_s\pi_i$ . Since  $W\Omega_{\mathbf{Q}}^n$  is qfh-separated for normal schemes, the sheaf condition for  $\pi'\pi$  follows from a verification of the sheaf condition for  $\pi_i$  and  $\pi_s$ , and this on the other hand induces the sheaf condition for  $\pi$ .

Suppose  $\pi$  is generically Galois with Galois group  $\text{Gal}(\kappa(V)/\kappa(X)) = \Gamma$ . By definition  $V$  is the normalization of  $X$  in  $\kappa(V)$  and the action of  $\Gamma$  extends by functoriality from  $\kappa(V)$  to  $V$ . Furthermore, the canonical map  $\mathcal{O}_X \xrightarrow{\sim} (\pi_*\mathcal{O}_V)$  is an isomorphism, which induces an isomorphism of Witt differentials. Hence we have an injection  $W\Omega_{\mathbf{Q}}^n(X) \cong W\Omega_{\mathbf{Q}}^n(V)^\Gamma \subset W\Omega_{\mathbf{Q}}^n(V)$ .

To show exactness at the second entry of the sequence, note that  $W\Omega_{\mathbf{Q}}^n$  is unramified and so the morphisms  $W\Omega_{\mathbf{Q}}^n(V) \hookrightarrow W\Omega_{\mathbf{Q}}^n(\kappa(V))$  and  $W\Omega_{\mathbf{Q}}^n(\tilde{V}) \hookrightarrow W\Omega_{\mathbf{Q}}^n(\kappa(\tilde{V}))$  are injective. Thus the kernel of  $W\Omega_{\mathbf{Q}}^n(V) \rightarrow W\Omega_{\mathbf{Q}}^n(\tilde{V})$  is contained in the kernel of  $W\Omega_{\mathbf{Q}}^n(\kappa(V)) \rightarrow W\Omega_{\mathbf{Q}}^n(\kappa(\tilde{V})) = W\Omega_{\mathbf{Q}}^n(\kappa(V) \otimes_{\kappa(X)} \kappa(V))$ . By Galois descent an element  $x$  in the kernel of  $W\Omega_{\mathbf{Q}}^n(V) \rightarrow W\Omega_{\mathbf{Q}}^n(\tilde{V})$  is then  $\Gamma$ -invariant, i.e.  $x \in W\Omega_{\mathbf{Q}}^n(\kappa(V))^\Gamma \cong W\Omega_{\mathbf{Q}}^n(X)$ .

Now we look at the case, when  $\pi$  is generically purely inseparable. By Zariski's connectedness theorem we know that since  $X$  is normal,  $\pi$  is actually purely inseparable, and hence its diagonal map is a surjective closed immersion. After reduction, we may assume that it is an isomorphism  $\Delta(\pi) : V^{\text{red}} \xrightarrow{\sim} (V \times_X V)^{\text{red}} = \tilde{V}$ . The map  $W\Omega_{\mathbf{Q}}^n(V) \rightarrow W\Omega_{\mathbf{Q}}^n(\tilde{V})$  being induced by the difference of the two projections  $\tilde{V} \rightarrow V$  consequently is the zero map, so that it remains to show that  $\pi^* : W\Omega_{\mathbf{Q}}^n(X) \hookrightarrow W\Omega_{\mathbf{Q}}^n(V)$  is surjective. If  $\pi$  is an isomorphism this is clear. Assume therefore that  $\pi$  is not an isomorphism. Similar to [AHL16] we reduce this to the case where  $\pi$  is a relative Frobenius as follows.

Denote by  $X^{(p)}$  the base change of  $X$  along the absolute Frobenius  $F_k$  of  $\mathrm{Spec}(k)$ . We have a commutative diagram

$$\begin{array}{ccccc}
 & & F_X & & \\
 & \swarrow & & \searrow & \\
 X & \xleftarrow{W} & X^{(p)} & \xleftarrow{F_{X/k}} & X \\
 \downarrow & & \downarrow & \nearrow & \\
 k & \xleftarrow{F_k} & k & & 
 \end{array}$$

where  $F_X$  is the absolute Frobenius of  $X$ ,  $F_{X/k}$  the relative Frobenius of  $X$  over  $k$  and  $W$  the canonical projection. From the hypothesis that  $F_k$  is an automorphism it follows that  $X \cong X^{(p)}$  and the relative Frobenius  $F_{X/k}$  coincides with the absolute Frobenius  $F_X$  up to isomorphism. For a  $p$ -power  $q = p^r$  we obtain a similar diagram by iteration.

By a result due to Kollár [Kol97, Prop. 6.6] there is a  $p$ -power  $q$  and a morphism  $\pi' : X \rightarrow V^{(q)}$  such that the relative Frobenius factors  $F_{X/k}^r = \pi' \circ \pi : X \rightarrow X^{(q)}$ . Thus it suffices to show that  $(F_{X/k}^r)^*$  is surjective.

Since  $X$  as an irreducible scheme is an  $\mathbf{F}_p$ -scheme, the Witt vector Frobenius is induced by the absolute Frobenius  $F_X^* = F : W\Omega^n(X) \rightarrow W\Omega^n(X)$ , which commutes with the Verschiebung map. In particular,  $FV = VF = p$ . Consequently, after inverting  $p$ ,  $F = F_X^* = F_{X/k}^*$ , as well as  $(F_{X/k}^r)^*$  for all  $q = p^r$ .  $\square$

**Remark 4.8.** One notes from the proof that  $W\Omega^n$  is a qfh-sheaf after inverting only the Witt vector Frobenius.

**Corollary 4.9.** The extension  $W\Omega_{\mathbf{Q}, \mathrm{dvr}}^n$  is a qfh-sheaf on  $\mathrm{Sch}(k)$ .

*Proof.* According to [HKK15, Lem. 4.4] the extension functor preserves sheaves. Hence the fact that  $W\Omega_{\mathbf{Q}}^n$  is a qfh-sheaf on  $\mathrm{Reg}(k)$  because it is one on  $\mathrm{Nor}(k)$  implies that  $W\Omega_{\mathbf{Q}, \mathrm{dvr}}^n$  is a qfh-sheaf on  $\mathrm{Sch}(k)$ .  $\square$

We will utilize the following lemma, which is well-known to experts.

**Lemma 4.10.** Let  $X \in \mathrm{Sch}(k)$ . Every h-cover  $\{U_i \xrightarrow{p_i} X\}_i$  has a refinement of the form

$$\{V_i \rightarrow \overline{V} \rightarrow X' \rightarrow X\}_i$$

where  $\{V_i \rightarrow \overline{V}\}_i$  is a Zariski cover,  $\overline{V} \rightarrow X'$  is a modification and  $X' \rightarrow X$  is a finite morphism.

*Proof.* By de Jong's alteration theorem [dJ96] choose a regular alteration  $Y \rightarrow X$  and consider the pullback  $\{U'_i \rightarrow Y\}$  of the original covering to  $Y$ . Let  $\{V_i \rightarrow \overline{V} \rightarrow Y_Z \rightarrow Y\}_i$  be a refinement of normal form of  $\{U'_i \rightarrow Y\}$  which exists by Voevodsky's Theorem 2.3, meaning that  $\{V_i \rightarrow \overline{V}\}$  is a Zariski cover,  $\overline{V} \rightarrow Y_Z$  is finite surjective and  $Y_Z \rightarrow Y$  is the blow-up along a closed subscheme  $Z$  of  $Y$ . We obtain a refinement of the original cover

$$\{V_i \rightarrow \overline{V} \rightarrow Y_Z \rightarrow Y \rightarrow X\}_i.$$

The composition  $\overline{V} \rightarrow Y_Z \rightarrow Y \rightarrow X$  is an alteration, therefore, it factors using Stein factorization  $\overline{V} \rightarrow X' \rightarrow X$ , where  $\overline{V} \rightarrow X'$  is proper with geometrically connected fibers and  $X' \rightarrow X$  is finite. This gives the desired refinement.  $\square$

We close this section with a descent result which we hope, it will be of independent interest.

**Theorem 4.11.** Fix a base scheme  $S$  and let  $\mathcal{F}$  be an unramified étale sheaf on  $\mathrm{Reg}(S)$ . When  $\mathcal{F}$  is also a qfh-sheaf, then  $\mathcal{F}_{\mathrm{dvr}}$  satisfies h-descent on  $\mathrm{Sch}(S)$ .

*Proof.* By hypothesis  $(\mathcal{F}_{\mathrm{dvr}})_{\mathrm{eh}} \cong \mathcal{F}_{\mathbf{Q}, \mathrm{dvr}}$  and  $(\mathcal{F}_{\mathrm{dvr}})_{\mathrm{qfh}} \cong \mathcal{F}_{\mathrm{dvr}}$ . It suffices to check that  $\mathcal{F}_{\mathrm{dvr}}$  agrees with its h-sheafification which may be checked via Čech cohomology. That is for any  $X \in \mathrm{Sch}(k)$ , the natural map

$$\mathcal{F}_{\mathrm{dvr}}(X) \rightarrow \lim_{\mathcal{U}} \check{H}^0(\mathcal{U}, \mathcal{F}_{\mathrm{dvr}})$$

where  $\mathcal{U}$  runs through all h-covers  $\mathcal{U} \rightarrow X$ , is an isomorphism.

Any h-cover  $\mathcal{U} \rightarrow X$  may be refined as  $\{W_i \rightarrow X' \rightarrow X\}_{i \in I}$  where the first is an eh-cover and the latter is a qfh-cover by Lemma 4.10. As  $\mathcal{F}_{\mathrm{dvr}}$  is both a qfh-sheaf and an eh-sheaf, the sheaf condition holds in the intermediate steps. More precisely, this means, that the sequences

$$0 \rightarrow \mathcal{F}_{\mathrm{dvr}}(X) \rightarrow \mathcal{F}_{\mathrm{dvr}}(X') \rightarrow \mathcal{F}_{\mathrm{dvr}}(X' \times_X X') \quad (4.2)$$

and

$$0 \rightarrow \mathcal{F}_{\mathrm{dvr}}(X') \rightarrow \prod_{i \in I} \mathcal{F}_{\mathrm{dvr}}(W_i) \rightarrow \prod_{i, j \in I} \mathcal{F}_{\mathrm{dvr}}(W_i \times_{X'} W_j) \quad (4.3)$$

are exact. These exact sequences clearly give injections

$$\mathcal{F}_{\text{dvr}}(X) \hookrightarrow \mathcal{F}_{\text{dvr}}(X') \hookrightarrow \prod_{i \in I} \mathcal{F}_{\text{dvr}}(W_i)$$

as well as

$$\mathcal{F}_{\text{dvr}}(X' \times_X X') \hookrightarrow \prod_{i,j \in I} \mathcal{F}_{\text{dvr}}(W_i \times_X W_j).$$

We obtain a commutative diagram

$$\begin{array}{ccccc} 0 \hookrightarrow \mathcal{F}_{\text{dvr}}(X) & \hookrightarrow & \mathcal{F}_{\text{dvr}}(X') & \hookrightarrow & \prod_{i \in I} \mathcal{F}_{\text{dvr}}(W_i) \\ & & \downarrow & & \downarrow \\ & & \mathcal{F}_{\text{dvr}}(X' \times_X X') & \hookrightarrow & \prod_{i,j \in I} \mathcal{F}_{\text{dvr}}(W_i \times_X W_j) \\ & & & & \downarrow \\ & & & & \prod_{i,j \in I} \mathcal{F}_{\text{dvr}}(W_i \times_{X'} W_j) \end{array}$$

With a diagram chase, keeping in mind the exact sequences (4.2) and (4.3), we obtain the exact sequence

$$0 \rightarrow \mathcal{F}_{\text{dvr}}(X) \rightarrow \prod_{i \in I} \mathcal{F}_{\text{dvr}}(W_i) \rightarrow \prod_{i,j \in I} \mathcal{F}_{\text{dvr}}(W_i \times_X W_j), \quad (4.4)$$

which is exactly the sheaf property for the cover  $\{W_i \rightarrow X' \rightarrow X\}_{i \in I}$ . This shows the claim.  $\square$

**Corollary 4.12.** *The sheaves  $W\Omega_{\mathbf{Q},\text{dvr}}^n$ ,  $n \geq 0$ , on  $\text{Sch}(k)$  satisfy h-descent.*

*Proof.* We have already seen that  $W\Omega_{\mathbf{Q}}^n$  is an unramified étale sheaf by Theorem 4.3 and by Theorem 4.9 a qfh-sheaf as well. Thus Theorem 4.11 applies.  $\square$

**4.3. Topological Torsion.** For a presheaf  $\mathcal{F}$  on  $\text{Sch}(S)$ , we denote as in [HKK15]  $\text{tor } \mathcal{F}(X)$  for the sections of  $\mathcal{F}(X)$  which vanish on a dense open subscheme and we call a presheaf  $\mathcal{F}$  torsion free when  $\text{tor } \mathcal{F}(X) = 0$ . After a general observation we look at the torsion forms for  $W\Omega_{\mathbf{Q}}^n$  over a perfect field of positive characteristic.

**Lemma 4.13.** *An unramified presheaf  $\mathcal{F}$  has no torsion forms on  $\text{Reg}(S)$ .*

*Proof.* Let  $\mathcal{F}$  be an unramified sheaf on  $\text{Reg}(S)$  and  $X \in \text{Reg}(S)$ . Suppose  $\omega \in \mathcal{F}$  is torsion. This means that there is an open dense subscheme  $U \hookrightarrow X$  such that  $\omega|_U = 0$ . However, the second characterizing property for unramified presheaves gives that the induced morphism  $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$  is injective. Thus  $\omega = 0$ .  $\square$

This means in particular that the sheaves  $W\Omega^n$  and  $W\Omega_{\mathbf{Q}}^n$  have no torsion forms on  $\text{Reg}(k)$ .

**Theorem 4.14.** *Let  $\mathcal{F}$  be an unramified presheaf and assume in addition that  $\mathcal{F}_{\text{dvr}}$  is an h-sheaf. Then  $\mathcal{F}_{\text{dvr}}$  is torsion free.*

*Proof.* Let  $X \in \text{Sch}(k)$  and  $\omega \in \mathcal{F}_{\text{dvr}}$  be a torsion form, meaning that there is an open dense subscheme  $U \hookrightarrow X$  such that  $\omega|_U = 0$ . We aim to show that  $\omega = 0$ . By definition  $\omega$  is given by a compatible system  $(\omega_Y)_Y$ , where  $Y \in \text{Reg}(X)$ . It suffices to show  $\omega_Y = 0$  for all such  $Y$ . For a morphism  $f : Y \rightarrow X$ ,  $Y \in \text{Reg}(X)$ , let  $Y_U$  be the preimage of  $U$  under  $f$ . The hypothesis  $\omega|_U = 0$  implies that  $\omega_Y|_{Y_U} := \omega_Y|_{Y_U} = 0$ .

If the preimage of  $U$  in  $Y$  under  $f$  is dense, it follows as in Lemma 4.13 that  $\omega_Y = 0$  because  $Y$  is regular. Otherwise, choose a regular alteration  $g : \tilde{X} \rightarrow X$ , i.e.  $\tilde{X} \in \text{Reg}(X)$ . The preimage  $\tilde{X}_U$  of  $U$  under  $g$  is again dense in  $\tilde{X}$ . The image of  $Y$  in  $X$  is contained in a subscheme  $Z$  where  $U \cap Z$  is not dense. Let  $\tilde{Z}$  be the pullback of  $Z$  along  $g$ . Furthermore, let  $\tilde{Y}$  be the pullback of  $Y$  along  $g$ , which is again regular. In particular,  $\tilde{Y} \rightarrow Y$  is as an alteration and thus an h-cover. We obtain a commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \tilde{Z} & \longrightarrow & Z \\ \downarrow & & \downarrow \\ \tilde{X} & \longrightarrow & X. \end{array}$$



By definition,  $\omega_{\tilde{X}} := \omega|_{\tilde{X}} \in \mathcal{F}(\tilde{X}) = \mathcal{F}_{\text{dvr}}(\tilde{X})$  constitutes an entry in the system  $(\omega_Y)_Y$ . Set  $\omega_{\tilde{Y}}$  the image of  $\omega_Y$  under the map  $\mathcal{F}(Y) \rightarrow \mathcal{F}(\tilde{Y})$ . We have the following commutative diagram where we have indicated where the sections  $\omega, \omega_Y, \omega_{\tilde{X}}$ , and  $\omega_{\tilde{Y}}$

$$\begin{array}{ccc} \omega_{\tilde{Y}} \in \mathcal{F}_{\text{dvr}}(\tilde{Y}) = \mathcal{F}(\tilde{Y}) & \longleftarrow & \mathcal{F}_{\text{dvr}}(Y) = \mathcal{F}(Y) \ni \omega_Y \\ \uparrow & & \uparrow \\ \omega_{\tilde{X}} \in \mathcal{F}_{\text{dvr}}(\tilde{X}) = \mathcal{F}(\tilde{X}) & \longleftarrow & \mathcal{F}_{\text{dvr}}(X) \ni \omega \end{array}$$

Considering that  $g^{-1}(U) = \tilde{X}_U$  is dense in  $\tilde{X}$  and  $\omega_{\tilde{X}}|_U = 0$ , we know that  $\omega_{\tilde{X}} = 0$  because  $\tilde{X}$  is regular. By compatibility the image  $\omega_{\tilde{Y}}$  of  $\omega_{\tilde{X}}$  under the morphism  $\mathcal{F}(\tilde{X}) \rightarrow \mathcal{F}(\tilde{Y})$  is also zero. We conclude that  $\omega_Y$  being in the preimage of  $\omega_{\tilde{Y}}$  under the map  $\mathcal{F}(Y) \rightarrow \mathcal{F}(\tilde{Y})$  is 0 as well, since  $\mathcal{F}_{\text{dvr}}^n$  is by assumption an h-sheaf and  $\tilde{Y} \rightarrow Y$  an h-cover.  $\square$

**Corollary 4.15.** *The sheaf  $W\Omega_{\mathbf{Q}, \text{dvr}}^n$  is torsion free on  $\text{Sch}(k)$ .*

*Proof.* As  $W\Omega_{\mathbf{Q}, \text{dvr}}^n$  is an h-sheaf Theorem 4.14 applies.  $\square$

## 5. PROPERTIES OF h-WITT DIFFERENTIALS

The critical property about the sheaves  $\Omega^n$  driving the many useful statements in [HJ14] is that  $\Omega_{\text{eh}}^n$  satisfies h-descent. Working in characteristic 0, one automatically gets h-descent for  $\Omega^n$  on smooth schemes following work of Geisser not available in our setting. Instead, we obtain h-descent for regular schemes from the fact that  $W\Omega_{\mathbf{Q}, \text{dvr}}^n$  is an h-sheaf.

**5.1. Regular descent.** As described in [HK15, Sec. 5], to understand  $\Omega_{\text{cdh}}^n$  it is critical to understand the torsion of  $\Omega^n$ . In our case, something similar is true: it is important to understand the torsion of  $W\Omega_{\mathbf{Q}}^n$  in order to understand  $W\Omega_{\mathbf{Q}, \text{h}}^n$ .

**Lemma 5.1.** *For a base scheme  $S$ , let  $\mathcal{F}$  be a sheaf on  $\text{Sch}(S)$  which is torsion free on  $\text{Reg}(S)$ . For  $X \in \text{Sch}(S)$  and  $\omega \in \text{tor } \mathcal{F}(X)$  there is an alteration  $\alpha: \tilde{X} \rightarrow X$  such that  $\alpha^*\omega$  vanishes in  $\mathcal{F}(\tilde{X})$ .*

*Moreover, if for each point  $x \in X$ ,  $\omega|_x = 0$ , there is an h-morphism  $\alpha: \tilde{X} \rightarrow X$  such that  $\alpha^*\omega = 0$ .*

*Proof.* As  $\omega$  is torsion, there is a Zariski open subset  $U \subset X$  such that the restriction  $\omega|_U = 0$ . If  $\alpha: \tilde{X} \rightarrow X$  is a regular alteration, then  $\alpha^*\omega \in \mathcal{F}(\tilde{X})$  vanishes on  $\alpha^{-1}U$ , which is dense open in  $\tilde{X}$ . But since  $\tilde{X}$  is regular,  $\mathcal{F}(\tilde{X})$  is torsion free by assumption and thus  $\alpha^*\omega = 0$ .

To prove the second claim, we may without loss of generality assume that  $X$  is integral by considering each irreducible component of  $X$  separately and using the fact that each scheme is eh-locally and therefore h-locally reduced. In this case, the assumption that  $\omega|_x = 0$  for all points  $x$  of  $X$ , means that  $\omega$  is torsion on  $X$  and we can now apply the first part to obtain an h-cover  $\alpha: \tilde{X} \rightarrow X$  with  $\alpha^*\omega = 0$ .  $\square$

**Definition 5.2.** Let  $\tau$  be a Grothendieck topology on  $\text{Sch}(S)$ . For a presheaf  $\mathcal{F}$  on  $\text{Reg}(S)$ , we say that it satisfies  $\tau$ -descent for regular schemes, if  $\mathcal{F}_\tau(X) = \mathcal{F}(X)$  for  $X \in \text{Reg}(k)$ . Equivalently we say that  $\mathcal{F}$  satisfies regular  $\tau$ -descent.

**Example 5.3.** When  $k$  is a field of characteristic 0,  $\mathcal{F} = \Omega^n$  satisfies regular h-descent by [HJ14, Thm. 3.6]. Over any field  $k$ , any sheaf  $\mathcal{F}$  for which  $\mathcal{F}_{\text{dvr}}$  which satisfies  $\tau$ -descent automatically satisfies regular  $\tau$ -descent.

We are ready to give the main descent theorem for the Witt differentials.

**Theorem 5.4.** *Let  $k$  be a perfect field of positive characteristic  $p$ . The sheaves  $W\Omega_{\mathbf{Q}}^n$ ,  $n \geq 0$ , on  $\text{Sch}(k)$  satisfy regular h-descent. In particular, when  $X$  is regular, we have isomorphisms  $W\Omega_{\mathbf{Q}, \text{h}}^n(X) \cong W\Omega_{\mathbf{Q}}^n(X)$ .*

*Proof.* Since  $W\Omega_{\mathbf{Q}, \text{dvr}}^n$  is an h-sheaf, we have by the universal property of sheafification a sequence of canonical morphisms

$$W\Omega_{\mathbf{Q}}^n \rightarrow W\Omega_{\mathbf{Q}, \text{h}}^n \rightarrow W\Omega_{\mathbf{Q}, \text{dvr}}^n. \quad (5.1)$$

The first step is to show, that the second map is a monomorphism. This follows a similar argument to [HK15, Prop. 5.12]. Let  $X \in \text{Sch}(k)$  and let  $\omega$  be a section in the kernel of  $W\Omega_{\mathbf{Q}, \text{h}}^n(X) \rightarrow W\Omega_{\mathbf{Q}, \text{dvr}}^n(X)$ . Choose an h-cover  $f: X' \rightarrow X$ , such that  $f^*\omega$  is in the image of  $W\Omega_{\mathbf{Q}}^n(X') \rightarrow W\Omega_{\mathbf{Q}, \text{h}}^n(X')$ . This way, we obtain an element  $\omega'$  in the kernel of  $W\Omega_{\mathbf{Q}}^n(X') \rightarrow W\Omega_{\mathbf{Q}, \text{dvr}}^n(X')$ , and have to show that it vanishes on some h-cover of  $X'$ . By Lemma 5.1, it is

enough to show that  $\omega'$  is trivial on every point  $x \in X'$ . For such a point, let  $\overline{\{x\}}$  be its closure in  $X'$  with reduced scheme structure, and let  $V$  be its regular locus. In particular,  $V \in \text{Reg}(X')$  and its generic point is isomorphic to  $x$ . Since  $\omega'$  is trivial in  $W\Omega_{\mathbf{Q}, \text{dvr}}^n(X')$  it vanishes by definition of the extension functor on every regular scheme over  $X'$ , specifically on  $V$  as well on its generic point and therefore on  $x$ .

For the second step, let  $X \in \text{Reg}(k)$ . Applying the global section functor with  $X$  to the composition of maps (5.1) we obtain a composition of natural maps  $W\Omega_{\mathbf{Q}}^n(X) \rightarrow W\Omega_{\mathbf{Q}, \text{h}}^n(X) \hookrightarrow W\Omega_{\mathbf{Q}, \text{dvr}}^n(X)$ , where the second morphism is injective by what we have just seen. By Remark 4.6,  $W\Omega_{\mathbf{Q}, \text{dvr}}^n(X) = W\Omega_{\mathbf{Q}}^n(X)$  and the above composition is actually an isomorphism, which induces that the second map is surjective, and therefore also an isomorphism, and it follows right away that the same is true for the first map.  $\square$

**Remark 5.5.** *In the case when  $n = 0$  there are stronger h-descent results known. Specifically, the sheaf  $W\mathcal{O}_{\mathbf{Q}}$  is an h-sheaf on  $\text{Sch}(k)$  by [BBE07, Thm. 2.4]. In fact, one even has cohomological descent for  $W\mathcal{O}$  for any finitely presented  $\mathbf{F}_p$ -scheme after just inverting the Witt vector Frobenius [BS16, Thm. 5.40]. This is unlike the situation for  $\Omega^0 = \mathcal{O}$  in characteristic 0, where one has such a result only for semi-normal schemes [HJ14, Prop. 4.5].*

It is well known to experts that h-sheaves can be understood as projective limits. The intuition is that sheaves which satisfy regular h-descent are h-locally smooth. Here a detailed proof of this fact is given and applied to Witt differentials.

**Lemma 5.6.** *Let  $\mathcal{F}$  be a sheaf on the Zariski site of  $\text{Sch}(S)$  which satisfies regular  $\tau$ -descent. Suppose  $X \in \text{Sch}(S)$ . Assume there is a regular  $\tau$ -cover  $X' \rightarrow X$  and a regular  $\tau$ -cover  $X'' \rightarrow X' \times_X X'$ . The sequence*

$$0 \rightarrow \mathcal{F}_{\tau}(X) \rightarrow \mathcal{F}(X') \rightarrow \mathcal{F}(X'')$$

*is exact.*

*Proof.* From the sheaf property, we have an exact sequence

$$0 \rightarrow \mathcal{F}_{\tau}(X) \rightarrow \mathcal{F}(X') \rightarrow \mathcal{F}(X' \times_X X')$$

for any  $\tau$ -cover  $X' \rightarrow X$ . For a  $\tau$ -cover  $X'' \rightarrow X' \times_X X'$ , the induced map  $\mathcal{F}_{\tau}(X' \times_X X') \hookrightarrow \mathcal{F}(X'')$  is again injective by the sheaf condition. Together, this gives an exact sequence  $0 \rightarrow \mathcal{F}_{\tau}(X) \rightarrow \mathcal{F}_{\tau}(X') \rightarrow \mathcal{F}_{\tau}(X'')$ . In the case when  $X'$  and  $X''$  are regular, by regular  $\tau$ -descent  $\mathcal{F}_{\tau}(X') = \mathcal{F}(X')$  and  $\mathcal{F}_{\tau}(X'') = \mathcal{F}(X'')$  thus the exact sequence becomes  $0 \rightarrow \mathcal{F}_{\tau}(X) \rightarrow \mathcal{F}(X') \rightarrow \mathcal{F}(X'')$ .  $\square$

One can now give the analogue of [HJ14, Thm. 1] in suitable generality.

**Theorem 5.7.** *Fix a base scheme  $S$ . For  $X \in \text{Sch}(S)$  and an unramified Zariski sheaf  $\mathcal{F}$  on  $\text{Reg}(S)$  satisfying regular h-descent, we have an isomorphism*

$$\mathcal{F}_{\text{h}}(X) \cong \varprojlim_{Y \in \text{Reg}(X)} \mathcal{F}(Y) = \left\{ (\alpha_f) \in \prod_{\substack{f: Y \rightarrow X \\ Y \text{ regular}}} \mathcal{F}(Y) \mid \forall \psi: \begin{array}{ccc} Y' & & \\ \psi \downarrow & \searrow f' & \\ Y & \xrightarrow{f} & X \end{array} \Rightarrow \psi^* \alpha_f = \alpha_{f'} \right\}$$

*Proof.* The proof follows the same steps as [HJ14, Cor. 3.9]. There is a natural injection  $\mathcal{F}_{\text{h}}(X) \hookrightarrow \varprojlim_{Y \in \text{Reg}(X)} \mathcal{F}(Y)$  as a section  $\beta \in \mathcal{F}_{\text{h}}(X)$  determines uniquely a compatible family  $\beta_f := f^* \beta \in \mathcal{F}_{\text{h}}(Y)$  with  $f: Y \rightarrow X$  and  $Y$  regular. By regular h-descent, it follows  $\beta_f \in \mathcal{F}_{\text{h}}(Y) = \mathcal{F}(Y)$ . The theorem follows by demonstrating an injection  $\varprojlim_{Y \in \text{Reg}(X)} \mathcal{F}(Y) \hookrightarrow \mathcal{F}_{\text{h}}(X)$ .

We choose smooth h-covers  $g: X' \rightarrow X$  and  $h: X'' \rightarrow X' \times_X X'$ , denote by  $i: X' \times_X X' \rightarrow X$  the canonical map and by  $\text{pr}_{1,2}: X' \times_X X' \rightarrow X'$  the projections, fitting in the diagram

$$\begin{array}{ccccc} X'' & & & & \\ & \searrow h & & & \\ & X' \times_X X' & \xrightarrow{\text{pr}_2} & X' & \\ & \downarrow \text{pr}_1 & \searrow i & \downarrow g & \\ & X' & \xrightarrow{g} & X & \end{array}$$

For a compatible family  $(\alpha_f)_f \in \varprojlim_{Y \in \text{Reg}(X)} \mathcal{F}(Y)$  we thus have

$$(\text{pr}_1 \circ h)^* \alpha_g = \alpha_{i \circ h} = (\text{pr}_2 \circ h)^* \alpha_g.$$

Consequently  $\beta \in \mathcal{F}_h(X)$  exists such that  $\alpha_g = g^* \beta$ . This gives us a morphism  $\varprojlim_{Y \in \text{Reg}(X)} \mathcal{F}(Y) \rightarrow \mathcal{F}_h(X)$ . It remains to check that this morphism is injective.

To check this, it suffices to show that for any morphism  $f : Y \rightarrow X$  with  $Y$  regular  $f^*(\beta) = \alpha_f$  holds. Let  $p : Y' \rightarrow Y \times_X X'$  be a smooth h-cover, denote  $j : Y \times_X X' \rightarrow X$  the canonical map and  $\text{pr}_Y : Y \times_X X' \rightarrow Y$  respectively  $\text{pr}_{X'} : Y \times_X X' \rightarrow X'$  the projections, which again fit in a diagram

$$\begin{array}{ccccc} Y' & & & & \\ & \searrow p & & & \\ & Y \times_X X' & \xrightarrow{\text{pr}_{X'}} & X' & \\ & \downarrow \text{pr}_Y & \searrow j & \downarrow g & \\ Y & \xrightarrow{f} & X & & \end{array}$$

We therefore have equalities

$$(\text{pr}_Y \circ p)^* \alpha_f = \alpha_{j \circ p} = (\text{pr}_{X'} \circ p)^* \alpha_g = (\text{pr}_{X'} \circ p)^* g^* \beta = (\text{pr}_Y \circ p)^* f^* \beta.$$

As  $\mathcal{F}$  is unramified, the map  $(\text{pr}_Y \circ p)^* : \mathcal{F}(Y) \rightarrow \mathcal{F}(Y)$  is injective, and we can conclude that  $\alpha_f = f^* \beta$ . This shows that the map  $\varprojlim_{Y \in \text{Reg}(X)} \mathcal{F}(Y) \rightarrow \mathcal{F}_h(X)$  is in fact injective and the theorem follows.  $\square$

Applying this general framework to Witt differentials results in the following corollaries. In particular, we will see that *dvr*-Witt differentials and h-Witt differentials coincide on  $\text{Sch}(k)$  for a perfect field  $k$  of positive characteristic  $p$ .

**Corollary 5.8.** *Given an object  $X$  in  $\text{Sch}(k)$ , and regular h-covers  $X' \rightarrow X$  and  $X'' \rightarrow X' \times_X X'$ , the following sequence is exact*

$$0 \rightarrow W\Omega_{\mathbf{Q},h}^n(X) \rightarrow W\Omega_{\mathbf{Q}}^n(X') \rightarrow W\Omega_{\mathbf{Q}}^n(X''). \quad (5.2)$$

*Proof.* It is enough to apply Lemma 5.6 with  $\mathcal{F} = W\Omega_{\mathbf{Q}}^n$ . This lemma applies as Theorem 5.4 gives the required regular h-descent.  $\square$

**Corollary 5.9.** *For  $X \in \text{Sch}(k)$  and each  $n \geq 0$ ,*

$$W\Omega_{\mathbf{Q},h}^n(X) \cong \varprojlim_{Y \in \text{Reg}(X)} W\Omega_{\mathbf{Q}}^n(Y).$$

*Proof.* For fixed  $n \geq 0$  the sheaf  $W\Omega_{\mathbf{Q}}^n$  is unramified by Theorem 4.3 and satisfies regular h-descent by Theorem 5.4. Now apply Theorem 5.7 with  $\mathcal{F} = W\Omega_{\mathbf{Q}}^n$ .  $\square$

**Corollary 5.10.** *For a Zariski sheaf  $\mathcal{F}$  on  $\text{Reg}(S)$ , which satisfies regular h-descent one has*

$$\mathcal{F}_{\text{dvr}}(X) = \varprojlim_{Y \in \text{Reg}(X)} \mathcal{F}(Y) \cong \mathcal{F}_h(X).$$

*In particular, for  $X \in \text{Sch}(k)$  and  $n \geq 0$  there is an isomorphism*

$$W\Omega_{\mathbf{Q},h}^n(X) \cong W\Omega_{\mathbf{Q},\text{dvr}}^n(X).$$

*Thus one may identify the sheaves  $W\Omega_{\mathbf{Q},h}^n$  and  $W\Omega_{\mathbf{Q},\text{dvr}}^n$  on  $\text{Sch}(k)$ .*

*Proof.* The first equality follows from the definition of the extension functor and the second isomorphism by Theorem 5.7. The other claims follow as  $W\Omega_{\mathbf{Q}}^n$  satisfies regular h descent by Theorem 5.4.  $\square$

**5.2. Fundamental properties and some examples.** After now establishing regular h-descent, we investigate some properties of h-Witt differentials in the spirit of [HJ14, Prop. 4.2]. We note that we cannot show a direct h-descent for eh-sheaves, thus our techniques differ mildly. Throughout this section, fix a perfect  $k$  of characteristic  $p > 0$ . For an h-sheaf  $F$  and object  $X$  in  $\text{Sch}(k)$ , we utilize the notation  $F|_X$  for the push-forward of  $F$  to the Zariski-site of  $X$ .

**Theorem 5.11.** *Let  $X \in \text{Sch}(k)$ . Set  $W\Omega_{\mathbf{Q},X}^{[n]} := j_* W\Omega_{\mathbf{Q},X^{\text{reg}}}^n$  where  $j : X^{\text{reg}} \rightarrow X$  is the inclusion of the regular locus. The sheaf  $W\Omega_{\mathbf{Q},h}^n|_X$  satisfies the following:*

- (i) *The sheaf  $W\Omega_{h,\mathbf{Q}}^n|_X$  is a quasi-coherent torsion free sheaf of  $W\mathcal{O}_{\mathbf{Q},X}$ -modules,*
- (ii) *If  $X$  is regular, then*

$$W\Omega_{\mathbf{Q},h}^n|_X = W\Omega_{\mathbf{Q},X}^n.$$

(iii) Let  $r : X_{\text{red}} \rightarrow X$  be the reduction. Then

$$r_* W\Omega_{\mathbf{Q},h}^n|_{X_{\text{red}}} = W\Omega_{\mathbf{Q},h}^n|_X.$$

(iv) If  $X$  is reduced, there is a natural inclusion

$$W\Omega_{\mathbf{Q},X}^n/\text{torsion} \subset W\Omega_{\mathbf{Q},h}^n|_X.$$

(v) Let  $X$  be normal and  $j : X^{\text{reg}} \rightarrow X$  the inclusion of the regular locus. There is a natural inclusion

$$W\Omega_{\mathbf{Q},h}^n \subset W\Omega_{\mathbf{Q},X}^{[n]}.$$

(vi) For  $n > \dim(X)$ ,  $W\Omega_{\mathbf{Q},h}^n|_X = 0$ .

*Proof.* As before we choose smooth h-covers  $g : X' \rightarrow X$  and  $h : X'' \rightarrow X' \times_X X'$ , denote  $i : X' \times_X X' \rightarrow X$  the canonical map and  $\tilde{h} = i \circ h$ . Without loss of generality, we assume that  $g$  and  $\tilde{h}$  are proper. Hence they are quasi-compact and quasi-separated. It follows that the direct image sheaves  $g_* W\Omega_{\mathbf{Q}}^n$  and  $\tilde{h}_* W\Omega_{\mathbf{Q}}^n$  are quasi-coherent, because this is the case for  $W\Omega_{\mathbf{Q}}^n$ . According to the exact sequence (5.2)  $W\Omega_{h,\mathbf{Q}}^n|_X$  is the kernel of a morphism  $g_* W\Omega_{\mathbf{Q}}^n \rightarrow \tilde{h}_* W\Omega_{\mathbf{Q}}^n$  of quasi-coherent sheaves, thus in itself quasi-coherent. Torsion freeness for  $W\Omega_{\mathbf{Q},\text{dvr}}^n = W\Omega_{\mathbf{Q},h}^n$  was already shown in Section 4.3. This completes (i).

Item (ii) follows immediately from regular h-descent in Theorem 5.4. Item (iii) is true as the inclusion of the reduced subscheme is an h-cover.

For (iv) consider the natural map  $W\Omega_{\mathbf{Q},X}^n \rightarrow W\Omega_{\mathbf{Q},h}^n|_X$ . For reason that  $W\Omega_{\mathbf{Q},h}^n|_X$  is torsion free by (i), we may quotient out torsion in the domain to get  $W\Omega_{\mathbf{Q},X}^n/\text{torsion} \rightarrow W\Omega_{\mathbf{Q},h}^n|_X$ . Let  $j : X^{\text{reg}} \hookrightarrow X$  be the regular locus. By the definition of torsion freeness we have  $W\Omega_{\mathbf{Q},h}^n|_X \subset j_* W\Omega_{\mathbf{Q},h}^n|_{X^{\text{reg}}} = j_* W\Omega_{\mathbf{Q},X^{\text{reg}}}^n$ , where the last equality holds because of (ii). Note that  $j_* W\Omega_{\mathbf{Q},X^{\text{reg}}}^n$  is torsion free and that it agrees with  $W\Omega_{\mathbf{Q},X}^n/\text{torsion}$  on  $X^{\text{reg}}$ . Hence  $W\Omega_{\mathbf{Q},X}^n/\text{torsion} \subset j_* W\Omega_{\mathbf{Q},X^{\text{reg}}}^n$  and the diagram

$$\begin{array}{ccc} W\Omega_{\mathbf{Q},h}^n|_X & \hookrightarrow & j_* W\Omega_{\mathbf{Q},X^{\text{reg}}}^n \\ \uparrow & \nearrow & \\ W\Omega_{\mathbf{Q},X}^n/\text{torsion} & & \end{array}$$

commutes, so that the vertical map is injective as well.

Now we consider  $X$  to be normal and let again  $j : X^{\text{reg}} \hookrightarrow X$  be its regular locus. Because  $W\Omega_{\mathbf{Q},h}^n|_X$  is torsion free, the restriction  $W\Omega_{\mathbf{Q},h}^n(X) \hookrightarrow W\Omega_{\mathbf{Q},h}^n(X^{\text{reg}}) = W\Omega_{\mathbf{Q}}^n(X^{\text{reg}})$  is an injection. By locality this is equally true for all open subschemes of  $X$ , so that we conclude (v).

For (vi), assume without loss of generality that  $X$  is reduced, and by torsion freeness restrict to an open subset where it is smooth. The vanishing of  $W\Omega_{\mathbf{Q},h}^n|_X$  for  $n > \dim(X)$  then follows from the vanishing of  $W\Omega_{\mathbf{Q}}^n$  for smooth schemes.  $\square$

We conclude by addressing expected results in mildly singular cases. We first record some facts, that will serve as useful tools. Recall that an abstract blow-up of a scheme  $X$  is the data  $(X', Z)$  where  $X'$  is a scheme,  $f : X' \rightarrow X$  is a proper map,  $Z \subset X$  a closed subscheme with preimage  $E \subset X'$ , and  $f$  induces an isomorphism  $X' \setminus E \xrightarrow{\sim} X \setminus Z$  outside  $Z$ . The eh-topology is generated by abstract blow-ups and étale coverings. As such, any abstract blow up is an h-cover as well and h-sheaves in some sense commute with these.

**Lemma 5.12.** *Let  $(X', Z)$  be an abstract blow-up of  $X \in \text{Sch}(k)$ . For an h-sheaf  $\mathcal{F}$  the induced square*

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(X') \\ \downarrow & & \downarrow \\ \mathcal{F}(Z) & \longrightarrow & \mathcal{F}(E) \end{array}$$

*is Cartesian.*

*Proof.* The proof does not differ from the one in characteristic 0 given in [HJ14, Prop. 3.3]. For a more detailed description see also the first part of the proof of [BS16, Prop. 2.8], where a more general situation is discussed.  $\square$

Thus one has an exact sequence

$$0 \rightarrow \mathcal{F}(X) \rightarrow \mathcal{F}(X') \oplus \mathcal{F}(Z) \rightarrow \mathcal{F}(E). \quad (5.3)$$

**Lemma 5.13.** *Let  $X$  be normal and irreducible,  $L/K(X)$  a Galois extension of the function field of  $X$  with Galois groups  $G$  and  $Y$  a normalization of  $X$  in  $L$ . Any étale sheaf  $\mathcal{F}$  with eh-descent has h-descent for  $f : Y \rightarrow X$ . More precisely,*

$$\mathcal{F}_h(X) \cong \mathcal{F}_h(Y)^G.$$

*Proof.* This is well-known to experts and can be proven with essentially the same argument as in [HJ14, Lem. 2.10]. In the case when  $X$  is smooth and  $f : Y \rightarrow X$  étale, the statement holds because  $\mathcal{F}$  is an étale sheaf and thus satisfies Galois descent. The general case follows by restricting to a smooth open subscheme of  $X$  where  $f$  is étale and perform the same diagram chase as in *loc. cit.*  $\square$

We next recover via Theorem 5.11 results analogous to [HJ14, Prop. 4.9] and [HJ14, Prop. 4.10] concerning normal crossing varieties and group quotients.

**Theorem 5.14.** *For  $X \in \text{Sch}(k)$  with normal crossings,*

$$W\Omega_{\mathbf{Q},h}^n|_X = W\Omega_{\mathbf{Q},X}^n/\text{torsion}.$$

*Proof.* The proof is a direct translation of [HJ14, Prop. 4.9] with their [HJ14, Prop. 4.2] replaced by Proposition 5.11. We repeat the argument for convenience of the reader.

By Proposition 5.11(iv) there is an inclusion

$$W\Omega_{\mathbf{Q},X}^n/\text{torsion} \subset W\Omega_{\mathbf{Q},h}^n|_X$$

because  $X$  is reduced. Working étale locally, one may assume that  $X$  is a union of smooth hyperplane sections and use induction with respect to the number of irreducible components  $c(X)$  of  $X$ .

If  $c(X) \leq 1$  then we are in the smooth case, and the statement is just Proposition 5.11(ii). Thus let  $c(X) > 1$ . Choose an irreducible component  $Z \subset X$ , let  $X' = \overline{X \setminus Z}$  and  $E$  the inverse image of  $Z$  under the map  $X' \rightarrow X$ . Then  $(X', Z)$  is an abstract blow-up of  $X$  and by Lemma 5.12 there is an exact sequence analogous to (5.3)

$$0 \rightarrow W\Omega_{\mathbf{Q},h}^n|_X \rightarrow W\Omega_{\mathbf{Q},h}^n|_{X'} \oplus W\Omega_{\mathbf{Q},h}^n|_Z \rightarrow W\Omega_{\mathbf{Q},h}^n|_E.$$

As in the classical case, we see that the pull-back of torsion free Witt differentials is again torsion free. Hence, the above sequence fits into a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & W\Omega_{\mathbf{Q},h}^n|_X & \longrightarrow & W\Omega_{\mathbf{Q},h}^n|_{X'} \oplus W\Omega_{\mathbf{Q},h}^n|_Z & \longrightarrow & W\Omega_{\mathbf{Q},h}^n|_E \\ & & \uparrow & & \parallel & & \uparrow \\ & & W\Omega_{\mathbf{Q},X}^n/\text{torsion} & \longrightarrow & W\Omega_{\mathbf{Q},X'}^n/\text{torsion} \oplus W\Omega_{\mathbf{Q},Z}^n/\text{torsion} & \longrightarrow & W\Omega_{\mathbf{Q},E}^n/\text{torsion} \end{array}$$

where the first and last vertical map are inclusions by (iv) of Proposition 5.11 and the middle vertical map is an isomorphism by (ii).

By local calculations the second line in the diagram is also exact. Thus a diagram chase shows that the first vertical map is an isomorphism as well.  $\square$

Next we look at quotient singularities, meaning singular varieties that can be expressed as quotients of a smooth variety by a finite group action. Although the associated projection is not a modification it is of course an alteration.

**Corollary 5.15.** *Let  $X \in \text{Reg}(k)$  be quasi-projective with an operation of a finite group  $G$ . One has*

$$W\Omega_{\mathbf{Q},h}^n|_{X/G} \cong (W\Omega_{\mathbf{Q},X}^n)^G.$$

*Proof.* The projection  $X \rightarrow X/G$  is ramified with Galois group  $G$ . Hence, the equivalence follows immediately from Galois descent in Lemma 5.13 and the fact that  $W\Omega_{\mathbf{Q}}^n$  is an étale sheaf with eh-descent.  $\square$

**Remark 5.16.** *Considering that  $X/G$  is normal there is according to Proposition 5.11(v) an inclusion*

$$W\Omega_{\mathbf{Q},h}^n|_{X/G} \subset W\Omega_{\mathbf{Q},X/G}^{[n]}.$$

*In analogy to the classical case, one expects this to be an isomorphism.*

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